Learning of Non-monotonic Target Functions by Ising Perceptrons

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ABSTRACT

We study supervised learning by Ising perceptrons for a class of non-monotonic teacher input-output relations employing the statistical mechanical method, that is, the replica method with the Gibbs algorithm. We consider up to one-step replica symmetry breaking (1RSB) solutions. Contrary to the monotonic cases in which the learning curve for a large number of training set is determined by a certain local property of the target relations [1,2,3] in the non-monotonic cases, the behaviour of the learning curve depends on the shape of the teacher input-output relation. In particular, for the class of teacher input-output relations, which represents an extension of the reversed-wedge problem studied by Kabashima and Inoue [4], a peculiar type of learning, perfect anti-learning(PAL), exists, whereas perfect learning (PL) [5,6] does not exist. In PL, the students' weight vectors coincide with the teacher's weight vector at a finite value of α, and in PAL the students' weight vectors coincide with the opposite of the teacher's weight vector at a finite value of α. Here, α is the ratio of the number of examples p to the number of synaptic weights N, $\alpha \equiv \frac{p}{N}$. The theoretical results are confirmed by the results of Monte Carlo simulations(MCS).

Keywords: Statistical Mechanics, Replica Method, Ising Perceptron, Perfect Learning, Perfect Anti-Learning.

1. FORMULATION

We consider a stochastic input-output relation, i.e. that of the teacher, between an N-dimensional input vector $\boldsymbol{x}$ and a binary output $r^o \in \{1, -1\}$, which is represented by a conditional probability $p_r(r^o|\boldsymbol{x})$. Input vectors $\boldsymbol{x}$ are normalized as $|\boldsymbol{x}| = \sqrt{N}$, and $p_r(r^o|\boldsymbol{x})$ is defined as a function of the inner product of the input $\boldsymbol{x}$ and the optimal weight $\boldsymbol{w^o}$ (the teacher's weight vector) as $p_r(+1|\boldsymbol{x}) = \mathcal{P}(u^o) = \frac{1+P(u^o)}{2}$ where $u^o = (\boldsymbol{x} \cdot \boldsymbol{w^o})/\sqrt{N}$. Here, $\mathcal{P}(u^o) \in [0, 1]$ represents the probability that u^o will yield an output of $+1$. Unlike the teacher, whose output is stochastic, the students are pure perceptrons; that is, an output r by a student with weight vector $\boldsymbol{w}$ resulting from an input $\boldsymbol{x}$ is given by $r = \mathrm{sgn}(u)$, where $u \equiv (\boldsymbol{x} \cdot \boldsymbol{w})/\sqrt{N}$. The components of their weight vectors $\boldsymbol{w^o}$ and $\boldsymbol{w}$ take the discrete values ± 1. Further,

$P(-u) = -P(u)$ is stipulated for simplicity. From each such $\boldsymbol{x}^{\mu}$, an output $r^o_{\mu}(= 1$ or $= -1)$ is obtained with the conditional probability $p_r(r^o_{\mu}|\boldsymbol{x}^{\mu})$. In this way, a set of p examples of input and teacher-generated output, $\xi_p = \{(\boldsymbol{x}^1, r^o_1), (\boldsymbol{x}^2, r^o_2), ..., (\boldsymbol{x}^p, r^o_p)\}$, which is called a training set for students, is generated. For a given realization of examples ξ_p, defining the 'energy' $E[w, \xi_p]$ of a student with weight vector $\boldsymbol{w}$ as the number of false predictions, the partition function Z with inverse temperature β is calculated by the replica method. Z depends on replica order parameters, i.e., the overlap of student weight vectors, $q^{ab} = \frac{(\boldsymbol{w}^a \cdot \boldsymbol{w}^b)}{N}$, its conjugate $\hat{q}^{ab}$, the overlap of the student weight vector and the optimal weight vector, $R^a = \frac{(\boldsymbol{w^o} \cdot \boldsymbol{w}^a)}{N}$, and its conjugate $\hat{R}^a$. (a and b are replica indices). In this paper, we consider mainly the case in which the optimal weight $\boldsymbol{w^o}$ yields an incorrect output for an input vector $\boldsymbol{x}$ with probability $1-k$ when $|u^o| \leq a$, where $0 \leq k < 1/2$. That is, we treat the following function

$$\mathcal{P}(u^o) = \begin{cases} 0 & (\text{ for } u^o < -a) \\ 1-k & (\text{ for } -a \leq u^o < 0), \\ k & (\text{ for } 0 \leq u^o < a) \\ 1 & (\text{ for } a \leq u^o) \end{cases}$$

where the value a defines the interval of values of u^o, $[-a, a]$, over which there is a finite probability that the teacher will return the wrong answer. In this study, first we consider the replica symmetric (RS) solution. When T is sufficiently small and α is sufficiently large, the entropy of the RS solution becomes negative and the RS solution becomes invalid. Then, next, we consider the one-step replica symmetry breaking (1RSB) solution.

2. RESULTS

We obtained the following results for $0 \leq k < 1/2$.

(1) In the system we study, PL does not exist, but PAL does exist.

(2) For the case $k = 0$, the behaviour of the learning curve can be classified into three cases, depending on the value of parameter a:

Region(A) $0 < a < a_{sp}$

For small α, the RS solution in branch I is valid and for large α, the 1RSB solution in branch I is valid. As $\alpha \to \infty$, $R(\alpha) \to R_+$ for the 1RSB solution, where in this case, R_+ (which always satisfies $0 < R_+ < 1$) is the value of R at which the generalization error ϵ_g and the free energy are minimal.

Region(B) $a_{sp} \leq a < a_c$

For small α, the RS solution in branch I is valid and for large α, the 1RSB solution in branch I is valid. Also, again here, as $\alpha \to \infty$, $R(\alpha) \to R_+$ for the 1RSB solution, where again in this case R_+ corresponds to a local minimum of ϵ_g and the free energy. In this case, the state corresponding to the minimum of the free energy is that of PAL.

Region(C) $a_c < a$. Fig.1

In this region, all solutions other than the PAL solution are continued to the finite range of α values. When T is large, only RS solutions exist, and when T is small, 1RSB solutions exist. There is a first-order transition from the RS solution in branch I or the 1RSB solution in branch I to the PAL solution as α is increased.

(3) For the case $0 < k < \frac{1}{2}$, results similar to those of the case $k = 0$ are obtained. Further, we found that as k increases, in the case that $R \to R_+$, the value of R_+ increases, and when the system jumps into the PAL state as α increases, the theoretical value of α_{max}, at which the transition takes place, and the value $\alpha^*(< \alpha_{max})$, at which the results obtained from the MCS begin to deviate from the theoretical curve, both increase. The increase of these values reflects very natural behaviour of the model: As the number of mistakes made by the teacher decreases, it becomes easier for the students to learn.

3. CONCLUSION

From the results obtained in this paper, we note that the asymptotic behaviour of learning curves depends on a and k as $\alpha \to \infty$. That is, for small a, $R \to R_+$ and for large a, $R \to -1$(PAL) when k is fixed. On the other hand, for small k, $R \to -1$ and for large k, $R \to R_+$ when a is fixed. Therefore, we conclude that, unlike in the case of monotonic teacher input-output relations [1,2,3], the nature of the learning exhibited by the students is not determined by the properties of $\mathcal{P}(u)$ around the decision boundary $u = 0$ but, rather, depends on the shape of $\mathcal{P}(u)$ over the entire range of values of u.

In this paper we considered the RS and 1RSB ansatz. With regard to the AT stability of the RS solution, according to our results, as α increases, the entropy of the RS solution in branch I becomes 0 before its AT-stability is lost. The numerical results indicate that the RS solution is valid at least as long as its entropy is positive. We did not calculate the AT-stability for the 1RSB solution, but we judged the validity of the 1RSB ansatz by comparing the theoretical results and the numerical results obtained from the MCS. Although there exist finite-size effects in the MCS, we found that the numerical results support our ansatz.

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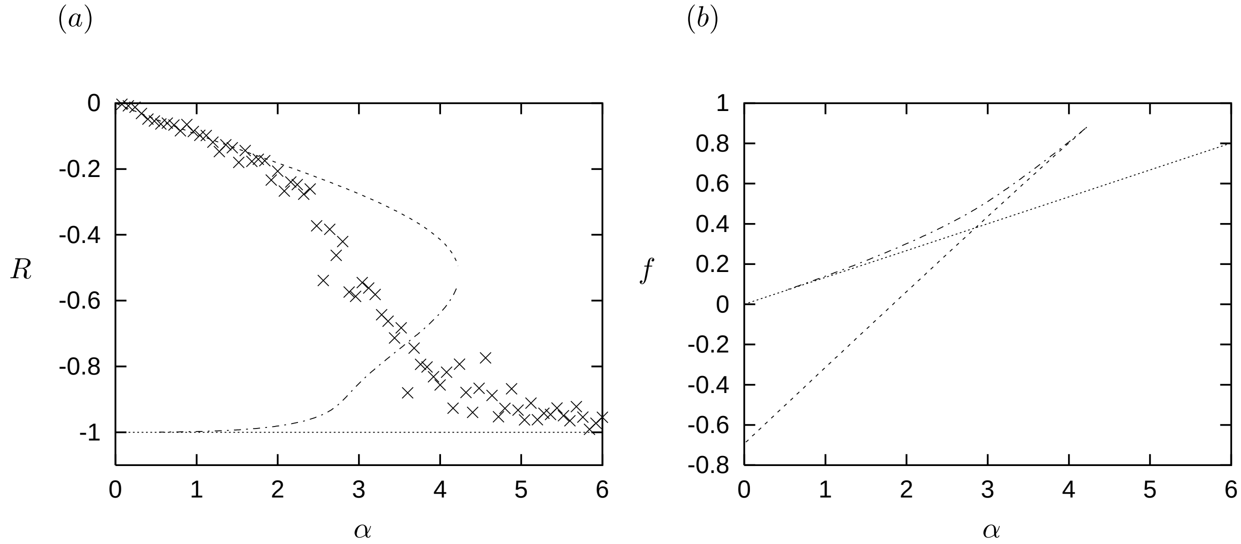

Fig. 1. Region (C). The α dependence of R and the free energy f for $k = 0$, $a = 1.5$, and $T = 1.0$. The dashed curve represents the RS solution in branch I, the dashed-dotted curve represents the RS solution in branch II, the dotted curve represents the PAL solution, and $\times$s represent the numerical results by MCS for $N = 25$. There is no RSB solution in this case.